J. Fluid Mech. (1991), *vol.* 223, *pp.* 497–508
Printed in Great Britain

Boundary conditions for rapid granular flows: phase interfaces

By J. T. JENKINS AND E. ASKARI
Department of Theoretical & Applied Mechanics, Cornell University, Ithaca, NY 14853, USA

(Received 31 October 1989 and in revised form 14 July 1990)

We consider the region of agitated grains that is at rest in the neighbourhood of its interface with a dense granular flow. We suppose that this region can be modelled as an amorphous solid of nearly elastic spheres in which both momentum and energy are transferred and energy is dissipated in collisions. Making rather rough assumptions about the collision probability, we calculate the stress and energy flux in the solid and use their continuity at the interface to obtain boundary conditions on the flow. We employ them with existing kinetic theory for nearly elastic spheres to solve boundary-value problems for shearing between two such interfaces and between such an interface and a flat plate to which spheres have been rigidly attached. For the latter, we compare the predictions of the theory with the results of experiments.

1. Introduction

Rapid flows of granular materials commonly occur in natural phenomena such as rock slides, debris flows, granular snow avalanches and underwater sediment slumps, and in industrial processes involving the rapid transport of bulk materials, for example, coal, ore, cereals, and pharmaceuticals. The common features in these diverse flows is that the grains interact with each other through collisions and that these collisions are responsible for the transfer of momentum and the transfer and dissipation of energy in the flow. Momentum and energy are supplied to a flow by gravity and by shear and normal forces applied at its boundaries. The interest, of course, is in predicting the relationship between the forces applied to a flow and its speed and extent.

Progress in this direction has recently been made for steady shearing flows of idealized materials maintained by the relative motion of equally idealized boundaries. Hanes, Jenkins and Richman (1988) study plane flows of dense collections of identical, smooth, nearly elastic circular disks driven by the relative motion of boundaries that consist of flat walls to which similar disks have been attached. They employ the balance laws and constitutive relations derived by Jenkins & Richman (1985) governing the fields of mean density, mean velocity and mean fluctuation energy, together with Richman & Chou's (1988) improvement of boundary conditions derived by Jenkins & Richman (1986) to solve the boundary-value problem for steady, inhomogeneous shearing between identical boundaries.

These solutions provide relations between the tangential and normal components of the traction applied to the boundaries, the distance between them, and their relative velocity. Numerical simulations of such shearing flows by Louge, Jenkins & Hopkins (1989) indicate that the predicted relations are very close to those measured in the simulations. In an Appendix, Hanes *et al.* (1988) sketch the derivation of the corresponding results for spheres. Jenkins & Askari (1990) show that the predictions for spheres are relative close to the experimental data of Craig, Buckholz & Domoto (1986) for the steady shear of steel spheres in an annular shear cell with carefully prepared upper and lower boundaries.

The success of the theory for steady shearing flows between boundaries to which the particles are rigidly attached, encourages us to extend it to other types of boundaries. Here we consider a dense aggregate of identical smooth spheres and focus on the interface between spheres that are being sheared and spheres that are, in mean, at rest. We treat this interface as the boundary between a dense gas and an amorphous solid. In the solid the velocity fluctuations persist, at least for some distance from the interface, but the density is too high to allow flow. Collisions between grains permit the transfer of momentum and a shear stress can be supported, provided that the radial distribution function for a colliding pair is anisotropic.

At the interface, particles may be incorporated into the flow; consequently, the mean velocity relative to the interface is zero. In the absence of slip, the energy flux at the interface is continuous. These are, essentially, the boundary conditions that are required. We employ them with the field equations to solve two boundary-value problems for steady shear flow. In the first, the shearing is maintained by the relative motion of two phase interfaces. This is a shear band in which collisions alone determine the thickness and mechanical behaviour. In the second, the material is sheared between a phase interface and a flat wall to which spheres are attached. This corresponds to those annular shear cell experiments of Hanes & Inman (1985) in which only the upper portion of the granular material was sheared. In each case we determine the relationships between the normal and tangential tractions applied to the boundaries, the distance between them, and their relative velocity.

We should emphasize that we consider only smooth spheres. Friction could be included in collisions between particles as it is, for example, in Jenkins' (1990) treatment of frictional spheres interacting with a flat frictional wall. At phase interfaces, friction in enduring contacts between particles may also be important. Johnson & Jackson (1987), for example, assume that at such an interface some particles are sliding and some particles are colliding. The tangential and normal tractions of the sliding particles are related by a Coulomb yield condition. Here, in the absence of any information regarding the relative importance of the various mechanisms for the transfer of momentum and energy at a phase interface, we propose and explore a model for what is probably the simplest of them.

2. Field equations

We consider rapid flows of a granular material consisting of identical smooth spheres of mass m and diameter σ. A coefficient of restitution, e, characterizes the energy lost when two spheres collide. We restrict our attention to nearly elastic collisions and employ standard results from kinetic theory (Chapman & Cowling 1970, chap. 16) slightly modified to include the collisional dissipation. The mean fields of interest are the mass density ρ, the product of m and the mean number n of spheres per unit volume; the mean velocity $\boldsymbol{u}$, about which the actual particle velocities fluctuate; and the granular temperature T, which measures the energy per unit mass of the velocity fluctuations.

The balance laws for these have the familiar local forms:

$$\dot{\rho}+\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \tag{1}$$

where an overdot indicates a time derivative following the mean motion;

$$\rho\dot{\boldsymbol{u}}=-\boldsymbol{\nabla}\cdot\boldsymbol{P}+n\boldsymbol{F}, \tag{2}$$

where $\boldsymbol{P}$ is the symmetric pressure tensor and $\boldsymbol{F}$ is the external force on a sphere; and

$$\tfrac{3}{2}\rho\dot{T}=-\boldsymbol{\nabla}\cdot\boldsymbol{Q}-\mathrm{tr}\,(\boldsymbol{P}\cdot\boldsymbol{\nabla}\boldsymbol{u})-\gamma, \tag{3}$$

where $\boldsymbol{Q}$ is the flux of fluctuation energy and γ is its rate of dissipation per unit volume in the inelastic collisions.

We focus attention on steady rectilinear flows in the (x,y)-plane in which the x-component u of the velocity, the density ρ, and the granular temperature T depend only on y. In this event, (1) is satisfied identically and, if external forces are assumed to be absent, the x- and y-components of (2) require that the shear stress $S\equiv -P_{xy}$ and the normal stress $N\equiv P_{yy}$ be constant. Then, with $Q\equiv Q_y$, (3) reduces to

$$Q'-Su'+\gamma=0, \tag{4}$$

where a prime denotes differentiation with respect to y.

The constitutive theory is based upon the assumption of binary collisions and Enskog's extension of the assumption of molecular chaos to dense systems. That is, the probability of a collision between a pair of spheres is assumed to be the product of the velocity distribution functions of each sphere, evaluated at its centre, and the radial distribution function for spheres in thermal equilibrium, evaluated at the midpoint of the line of centres of a colliding pair. Numerical simulations show that this value, g_0, of the radial distribution function depends upon the solid volume fraction $\nu\equiv\frac{1}{6}\pi n\sigma^3$; the observed dependence is well fitted by a function proposed by Carnahan & Starling (1969):

$$g_0(\nu)=\frac{2-\nu}{(1-\nu)^3}. \tag{5}$$

Here we restrict our attention to dense flows. In this case, the contributions from particle transport to the fluxes of momentum and energy are negligible. In addition, we retain only those contributions to the collisional fluxes that dominate in the dense limit. If we ignore any complications arising from correlated collisions at these densities (e.g. Dorfman & Kirkpatrick 1986), the normal stress and the shear stress may be written as the high-volume-fraction limits of expressions provided by Chapman & Cowling (1970, §16.41):

$$N=\pi^{\frac{1}{2}}\kappa T^{\frac{1}{2}}/\sigma, \tag{6}$$

where

$$\kappa\equiv(4/\pi^{\frac{1}{2}})\rho\sigma T^{\frac{1}{2}}G \tag{7}$$

with $G\equiv\nu g_0$, and

$$S=\tfrac{2}{5}JKu', \tag{8}$$

where $J\equiv 1+\frac{1}{12}\pi$. Upon eliminating κ between (6) and (8), we obtain a simple relation between the velocity gradient and the temperature:

$$u'=\frac{5\pi^{\frac{1}{2}}}{2J}\frac{T^{\frac{1}{2}}}{\sigma}\frac{S}{N}. \tag{9}$$

Similarly, the energy flux (Chapman & Cowling 1970, §16.42) is, in the limit, given by

$$Q=-M\kappa T', \tag{10}$$

where $M \equiv 1+\frac{9}{32}\pi$. The dissipation rate per unit volume is (Jenkins & Savage 1983)

$$\gamma = 6\kappa(1-e)\,T/\sigma^2. \tag{11}$$

We employ (9), (10), and (11) in the energy balance (4) and replace κ by $\sigma N/(\pi T)^{\frac{1}{2}}$ wherever it occurs. The resulting equation, written in terms of $w \equiv T^{\frac{1}{2}}$, is

$$\sigma^2 w'' - k^2 w = 0, \tag{12}$$

where
$$k^2 \equiv [3(1-e)-(5\pi/4J)\,(S/N)^2]/M. \tag{13}$$

When k is real, the solution of (12) involves hyperbolic functions; when k is imaginary, it involves trigonometric functions. However, such steady solutions are possible only if the boundary conditions permit them (see e.g. Jenkins & Richman 1986 and Hanes *et al.* 1988).

3. Phase interfaces

Often bounding such a rapid flow is a region in which the mean velocity vanishes but collisions between particles may persist, at least for some distance away from the flow. In this region the fluctuation energy supplied at the interface on which the mean velocity vanishes is conducted away from this surface, dissipated in collisions, and eventually disappears.

We suppose that in the region in which the mean flow vanishes but the collisions persist, the material has the structure of an amorphous, or glassy, solid. This is consistent with the observation of a glass transition at $\nu = 0.57$ in numerical simulations of elastic spheres (Woodcock 1981). In the solid, the volume fraction is too high to permit flow, but the stress continues to result from the transfer of momentum in collisions. However, the shear stress is due to an anisotropy in the radial distribution function induced by the distortion of the solid, rather than to the anisotropy in the velocity distribution functions associated with the velocity gradients.

In the solid, the modified radial distribution function, g, for a colliding pair may be written in terms of a deviatoric tensor $\boldsymbol{e}$ that characterizes the anisotropy and the unit vector $\boldsymbol{k}$ along the line of centres:

$$g = g_0(1-k_i\,e_{ij}\,k_j). \tag{14}$$

If we make the crude assumptions that molecular chaos prevails in the solid and the single-particle velocity distribution function is nearly Maxwellian, the constitutive relations may be obtained from the results of an analogous calculation by Jenkins & Savage (1983) that involves a radial distribution function distorted by the flow. The expressions for the normal stress and volume dissipation rate are, respectively, identical to (6) and (11), the energy flux is given by (10), and

$$S = \tfrac{2}{5} N e_{xy}. \tag{15}$$

Then, because the mean velocity vanishes in the glassy solid, w there is a solution of

$$\sigma^2 w'' - [3(1-e)/M]\,w = 0. \tag{16}$$

Solving this in the half-space $y \leqslant 0$ and discarding the unbounded part of the solution, we find that

$$w = w_0 \exp\{[3(1-e)]^{\frac{1}{2}} y/M^{\frac{1}{2}}\sigma\}, \tag{17}$$

where w_0 is the value of w at the interface. So, for nearly elastic spheres, the glassy solid has a thickness of several particle diameters. Within this layer ν must increase away from the interface in order to maintain constant S and N as the fluctuations diminish. Finally, the volume fraction becomes sufficiently high that the collisional momentum supplied to a particle over one portion of its surface is balanced by forces transmitted through enduring contacts over the other.

By our definition of the interface, the mean velocity of the flow relative to it vanishes; this is the boundary condition for (9). At the interface the fluctuation energy is continuous and, in the absence of a rate of working associated with a slip velocity, so is the energy flux. Consequently, the energy flux in the flow at the interface is, upon differentiating (17) and employing (10),

$$Q = \frac{-2\kappa M[3(1-e)]^{\frac{1}{2}} w_0^2}{M^{\frac{1}{2}}\sigma}. \tag{18}$$

The corresponding condition on the solution of (12) is

$$w' = \frac{[3(1-e)]^{\frac{1}{2}} w_0}{M^{\frac{1}{2}}\sigma}. \tag{19}$$

The fluctuation energy is continuous across the interface; so, in order to provide continuity of the normal traction, the solid volume fraction must also be continuous. Finally, at the interface the strain in the glassy solid must adjust to provide continuity of the tangential traction.

4. Shear bands

Shear bands are localized regions of intense shearing in granular materials. We investigate the possibility that at some stage in the development of such a band, the forces on either side of it are transmitted across it by collisions between particles. In order to study the thickness of the band and how the components of the transmitted force may be related to its motion, we consider the steady shearing flow established between two parallel phase interfaces separated by a distance L and moving relative to each other with a constant velocity $2U$. Here it is convenient to take the origin midway between the interfaces and to translate so that the interfaces move with the same speed U in opposite directions.

Symmetry then requires that $w'(0) = 0$, so the solution of (12) is

$$w = w_0 \frac{\cosh(ky/\sigma)}{\cosh(kL/2\sigma)}, \tag{20}$$

where here $w_0 \equiv w(-\frac{1}{2}L)$. Applying the boundary condition (19) at $y = -\frac{1}{2}L$, we obtain

$$-\tanh\left(\frac{kL}{2\sigma}\right) = \frac{[3(1-e)]^{\frac{1}{2}}}{kM^{\frac{1}{2}}}. \tag{21}$$

When k is real, this has no real solution for L. Consequently, for steady solutions to exist, it is necessary that

$$K^2 \equiv \frac{(5\pi/4J)(S/N)^2 - 3(1-e)}{M} > 0. \tag{22}$$

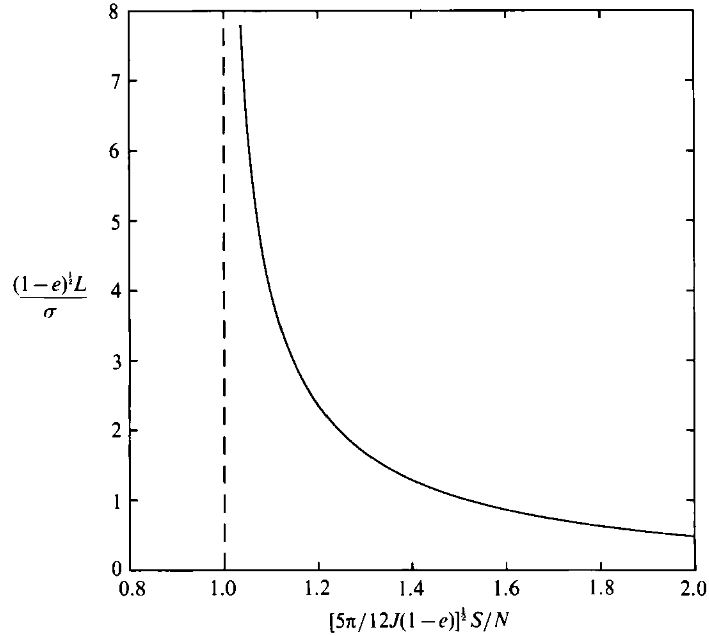

FIGURE 1. Normalized band thickness versus normalized stress ratio.

In this case, with $k \equiv \mathrm{i}K$, equation (21) becomes

$$\tan\left(\frac{KL}{2\sigma}\right) = \frac{[3(1-e)]^{\frac{1}{2}}}{KM^{\frac{1}{2}}}. \tag{23}$$

This determines L/σ as a function of K or, equivalently $(1-e)^{\frac{1}{2}}L/\sigma$ as a function of $[5\pi/12J(1-e)]^{\frac{1}{2}}S/N$. The graph of this relationship is given in figure 1 for the principal value of the argument. Determinations for other than the principal value lead to negative temperatures in the flow and must be discarded. The vertical asymptote corresponds to a homogeneous shearing with a linear velocity profile and uniform fluctuation energy and volume fraction. As the stress ratio increases, the thickness of the band decreases.

With (20) written in terms of K and the symmetry condition $u(0) = 0$, the solution of (9) for the mean velocity is

$$\frac{u}{w_0} = \frac{5^{\frac{1}{2}}\pi}{2J}\frac{S}{N}\frac{\sin(Ky/\sigma)}{K\cos(KL/2\sigma)}. \tag{24}$$

As a consequence of there being no slip on the interface, we have $u(-\frac{1}{2}L) = -U$. Hence

$$\frac{u}{w_0} = \frac{5\pi^{\frac{1}{2}}}{2J}\frac{S}{N}\frac{\tan(KL/2\sigma)}{K}; \tag{25}$$

or, with (23),

$$\frac{U}{w_0} = \frac{5\pi^{\frac{1}{2}}}{2J}\frac{S}{N}\frac{[3(1-e)]^{\frac{1}{2}}}{K^2M^{\frac{1}{2}}}. \tag{26}$$

With (22), this may be written as a quadratic equation for the stress ratio that, when solved, yields

$$\frac{S}{N} = \left[\frac{3(1-e)M}{\pi}\right]^{\frac{1}{2}}\left\{\frac{w_0}{U} + \left[\left(\frac{w_0}{U}\right)^2 + \frac{4J}{5M}\right]^{\frac{1}{2}}\right\}. \tag{27}$$

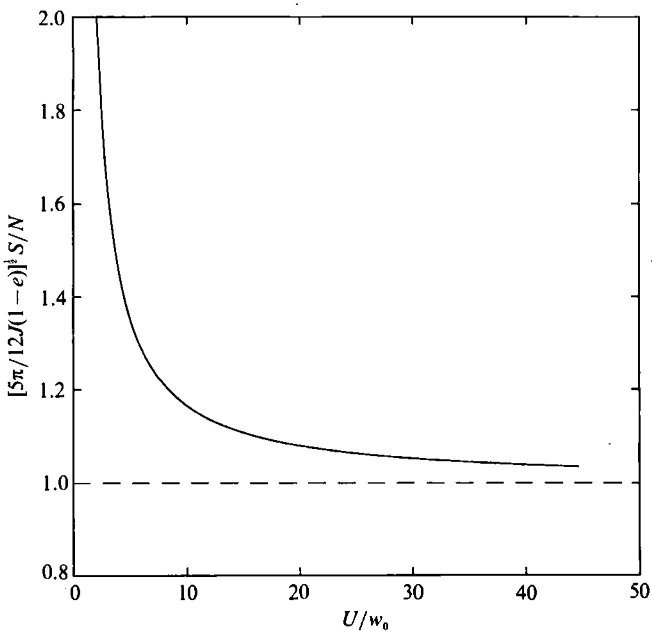

FIGURE 2. Normalized stress ratio versus normalized boundary velocity.

The graph of U/w_0 versus $[5\pi/12J(1-e)]^{\frac{1}{2}}S/N$ is given in figure 2. Here the horizontal asymptote corresponds to homogeneous shearing. The stress ratio is a monotonic decreasing function of normalized relative velocity.

Finally, using (6), we have $w_0^2 = N/4\rho_0 G_0$, where $\rho_0 = 6m\nu_0/\pi\sigma^3$. This permits the stress ratio to be expressed as a function of the boundary velocity, the normal traction, the mass and diameter of the spheres, and the solid volume fraction at the phase interface. The numerical simulations of Woodcock (1981) suggest that the value of ν_0 is between 0.57 and 0.64.

5. Depth of shear

We next consider steady shearing flow between a moving plate with a bumpy surface and a phase interface. We are particularly interested in the depth of material that participates in the flow. We suppose that the plate is made bumpy by randomly affixing to it spheres of diameter d with a mean spacing s between their nearest points. Boundary conditions at such a boundary have been given by Jenkins (1987) and Richman (1988) with the same accuracy as the constitutive relations (6)–(11). Hanes *et al.* (1988) employ them to study the steady symmetric shearing of identical spheres between two identical boundaries that are in relative motion. Our flow is asymmetric, so we take the phase interface to be at $y = 0$ with the bumpy boundary at $y = L$ moving parallel to it with velocity $2U$ in the x-direction.

At $y = L$, the rate, $\boldsymbol{M}$, at which momentum is supplied to the flow through collisions over a unit area of the wall has components

$$M_y = -\rho\chi w_1^2, \tag{28}$$

where χ is a function of the solid fraction that accounts for the effects of particle shielding and excluded area on the frequency of boundary collisions, and $w_1 \equiv w(L)$; and

$$M_x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \rho\chi w_1 \left\{\left[\frac{2(2-3\cos\theta+\cos^3\theta)(v-\bar{\sigma}u')}{3\sin^2\theta}\right] + (\tfrac{1}{2}\bar{\sigma}u')\left(1+\frac{\sigma\pi}{\bar{\sigma}12\surd 2}\right)\sin^2\theta\right\}, \quad (29)$$

where $v \equiv 2U - u(L)$ is the slip velocity, $\bar{\sigma} \equiv \frac{1}{2}(d+\sigma)$, and $\sin\theta \equiv (d+s)/(d+\sigma)$.

Balance of momentum at the boundary requires that $-M_y = N$ and $M_x = S$. The first of these used with (28) gives

$$N = \rho\chi w_1^2. \quad (30)$$

The second may be rewritten by first employing (9) and (30) to eliminate, respectively, u' and χ from the expression (29) for M_x. Then, upon solving for v/w_1 we obtain

$$\frac{v}{w_1} = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{S}{N} f_1, \quad (31)$$

where

$$f_1 \equiv \frac{3\sin^2\theta\{1-(5\surd 2/4J)(\bar{\sigma}/\sigma)[1+(\pi/12\surd 2)(\sigma/\bar{\sigma})\sin^2\theta]\}}{2(2-3\cos\theta+\cos^3\theta)} + \frac{5}{J\surd 2}\frac{\bar{\sigma}}{\sigma}. \quad (32)$$

The rate, D, of collisional dissipation per unit area of the boundary is

$$D = (2/\pi)^{\frac{1}{2}} 2\rho\chi(1-\epsilon)\, w^3(1-\cos\theta)\operatorname{cosec}^2\theta, \quad (33)$$

where ϵ is the coefficient of restitution of a collision between a flow sphere and a wall sphere. Balance of energy at the wall requires that $Sv - D = Q$. In this we use the constitutive relation (10) for Q and equation (31) for v/w_1. Upon replacing κ by $\sigma N/\pi^{\frac{1}{2}} w_1$ in the result, we obtain

$$\sigma w'/w_1 = b_1, \quad (34)$$

where

$$b_1 \equiv -\frac{[2(1-\epsilon)(1-\cos\theta)\operatorname{cosec}^2\theta - (\tfrac{1}{2}\pi f_1)(S/N)^2]}{M\surd 2}. \quad (35)$$

At the phase interface, $y = 0$, the boundary conditions are

$$v = 0 \quad (36)$$

and

$$\sigma w'/w_0 = -b_0, \quad (37)$$

where $w_0 \equiv w(0)$ and, from (19),

$$b_0 = -[3(1-e)/M]^{\frac{1}{2}}. \quad (38)$$

When k is real, the homogeneous equation (12) has a solution that satisfies the homogeneous boundary conditions (34) and (37) only if

$$\tanh\left(\frac{kL}{\sigma}\right) = \frac{(b_0+b_1)k}{b_0 b_1 + k^2}. \quad (39)$$

When $K \equiv -ik$ is real, the corresponding condition is

$$\tan\left(\frac{KL}{\sigma}\right) = \frac{(b_0+b_1)K}{b_0 b_1 - K^2}. \quad (40)$$

In figure 3 we plot the relationship between L/σ and S/N for values of $e, \epsilon, \bar{\sigma}$, and θ corresponding to our best estimates of these in the experiments of Hanes & Inman

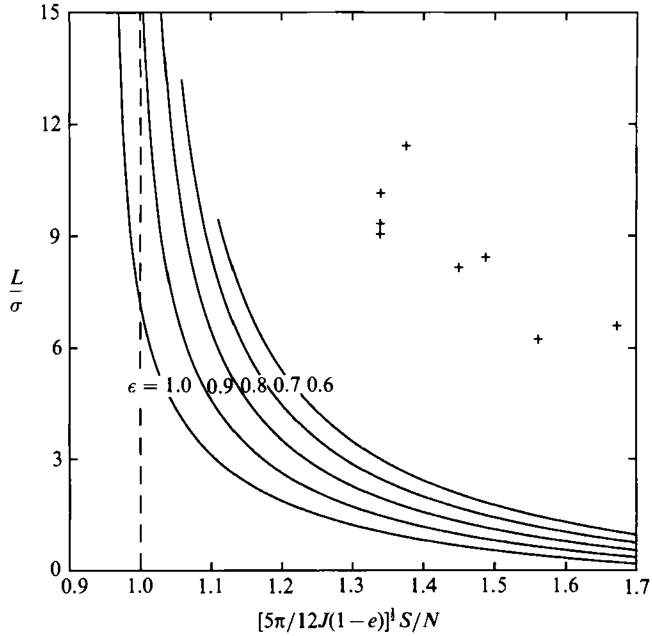

FIGURE 3. Normalized depth of shear versus normalized stress ratio and the data of Hanes & Inman (+). $d = \sigma$, $s/d = 0.75$, $e = 0.925$ and $\nu_0 = 0.06$.

(1985). To the left of the vertical dashed line k is real, fluctuation energy is dissipated in the interior, and the solution of (12) is a linear combination of $\sinh(ky/\sigma)$ and $\cosh(ky/\sigma)$. At the intercept with the vertical dashed line $k = 0$ and the profile of w is linear. To the right of the vertical dashed line K is real, fluctuation energy is produced in the interior, and the solution of (12) is a linear combination of $\sin(Ky/\sigma)$ and $\cos(Ky/\sigma)$. When b_1 is negative, the curves for K real terminate at $L/\sigma = \pi/2K$ as $b_0 b_1$ approaches K^2. As w varies across the thickness, the volume fraction must compensate in order to maintain a constant normal stress. In some situations, the change in volume fraction across the cell may be so great as to invalidate our restriction to dense flows. In this case we anticipate that our analysis captures most of the features of the more complicated problem. We note that the numerical simulations of Louge *et al.* (1989) indicate that the corresponding theory for plane flow of disks between identical bumpy boundaries applies to gaps as small as three diameters in width.

The points shown in figure 3 are data from Hanes & Inman (1985) for 1.1 mm glass spheres at volume fractions of 0.60 rapidly sheared through a part L of the depth of an annular cell with boundaries to which 1.1 mm glass spheres had been affixed. Only data for runs in which the weight of the material was a small fraction of the normal force on the upper boundary are shown. These data should all fall upon a single curve. There is scatter, but the data do seem to exhibit the same trend as the predictions. However, at a fixed value of L/σ, the predicted value of S/N is too low. We defer consideration of the reasons for this until after we have introduced a second parameterization of the problem.

When (39) is satisfied and a steady solution for w exists, its boundary values are related by

$$\frac{w_1}{w_0} = \left[\frac{k^2 - b_0^2}{k^2 - b_1^2}\right]^{\frac{1}{2}}. \tag{41}$$

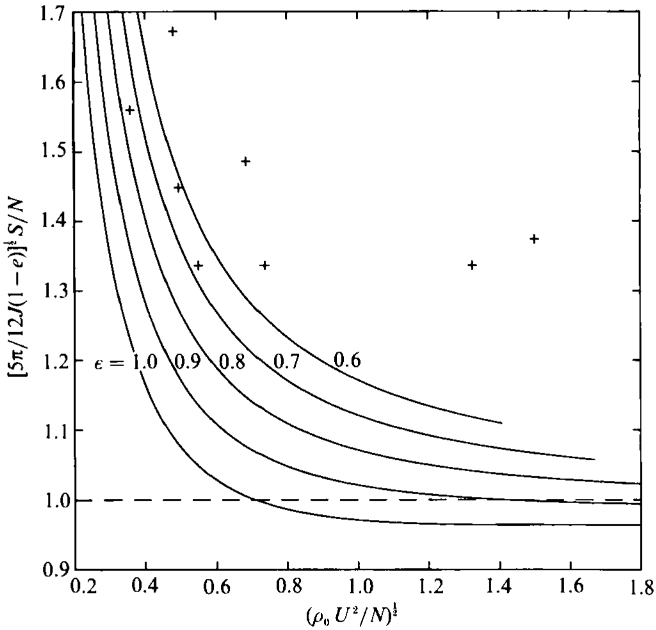

FIGURE 4. Normalized stress ratio versus normalized boundary velocity and the data of Hanes & Inman (+). Same parameters as in figure 3.

When (40) is satisfied, the corresponding condition is obtained by using $k = \mathrm{i}K$ in (41).

The second parameterization may be obtained by integrating (9) for the velocity. When this is done, the boundary conditions (31) and (36) are employed, and (39) and (41) are used to simplify the result, we obtain a relation between the boundary velocity and the stress ratio,

$$\frac{U}{w_0} = \pi^{\frac{1}{2}}\frac{S}{N}\left[\frac{5b_0}{4Jk^2}+\left[\frac{5b_1}{4Jk^2}+\frac{f_1}{2\surd 2}\right]\left[\frac{k^2-b_0^2}{k^2-b_1^2}\right]^{\frac{1}{2}}\right]. \tag{42}$$

After using (13), (32), (35), and (38) to make the dependence upon stress ratio, roughness, and the coefficients explicit and writing $w_0 = (N/4\rho_0 G_0)^{\frac{1}{2}}$, we plot this relation in figure 4 for the same values of $e, \epsilon, \bar{\sigma}$, and θ as in figure 3. We see that the stress ratio is a monotone decreasing function of normalized boundary velocity and, when fluctuation energy is being consumed in the interior and provided by the bumpy boundary, the stress ratio is less than its value in homogeneous shear. The points are the same data employed in figure 3. Again the data seem to exhibit the trend of the predictions, but the observed values of the stress ratio are higher than predicted.

As discussed by Jenkins & Askari (1990), the experimental system differs from that upon which the predictions are based in two important respects, the presence of sidewalls and the existence of friction. In the experiments, the bottom plate, including the sidewalls, rotates. Consequently, the mean velocity of particles near the wall is different from those in the centre, and the shear stress applied to the top of the cell must balance the additional momentum input from the sidewalls. Also, there is friction at the sidewalls and between particles, so there is an additional source of dissipation in the shear cell. Consequently, in the experiments higher shear stresses than predicted are required to produce the velocity fluctuations corresponding to a given normal stress. This is seen in the numerical simulations of simple shear in dense collections of frictional spheres by Walton (1990). For experiments in a cell 40 diameters wide these additional contributions to the shear stress are expected to be small but not negligible. If the experimental points are shifted to the left in figure 3 and down in figure 4 to correct for these differences between theory and experiment, the agreement can be made far more striking.

We must say that we have encountered difficulties in applying the theory for identical bumpy boundaries to the data of Hanes & Inman (1985) for dense aggregates that are sheared throughout their depth. Their several data points for 1.1 mm spheres at high volume fraction that are not influenced by gravity involve stress ratios that are about twice that predicted for the values of e, d, s, and ϵ used here. If we were to put our faith in the theories, the indication is that there were problems with the bottom boundary in the experiment. For example, the attached spheres may have been partially buried in glass dust.

In any case, we believe that we have provided a context in which granular flows bounded by regions of grains at rest may be placed. We look forward to additional physical experiments and numerical simulations that will highlight its deficiencies and suggest its improvement.

This work was partially supported by the US Army through the Mathematical Sciences Institute at Cornell University.

REFERENCES

CARNAHAN, N. F. & STARLING, K. E. 1979 Equations of state for non-attracting rigid spheres. *J. Chem. Phys.* **51**, 635–636.

CHAPMAN, S. & COWLING, T. G. 1970 *The Mathematical Theory of Nonuniform Gases*, 3rd edn. Cambridge University Press.

CRAIG, K. C., BUCKHOLZ, R. & DOMOTO, G. 1986 An experimental study of the rapid flow of dry cohesionless metal powders. *Trans. ASME* E: *J. Appl. Mech.* **53**, 935–952.

DORFMAN, J. R. & KIRKPATRICK, T. R. 1986 Recent results in the kinetic theory of hard sphere systems. In *Molecular Dynamics Simulations of Statistical–Mechanical Systems* (ed. G. Ciccotti & W. G. Hoover), pp. 260–293. North Holland.

HANES, D. M. & INMAN, D. L. 1985 Observations of rapidly flowing granular-fluid materials. *J. Fluid Mech.* **150**, 357–380.

HANES, D. M., JENKINS, J. T. & RICHMAN, M. W. 1988 The thickness of steady plane shear flows of circular disks driven by identical boundaries. *Trans. ASME* E: *J. Appl. Mech.* **55**, 969–974.

JENKINS, J. T. 1987 Rapid flows of granular materials. In *Non-Classical Continuum Mechanics* (R. J. Knops & A. A. Lacey), pp. 213–225. Cambridge University Press.

JENKINS, J. T. 1990 Boundary conditions for rapid granular flow: flat, frictional walls. *Trans. ASME* E: *J. Appl. Mech* (under review).

JENKINS, J. T. & ASKARI, E. 1990 Comparison of theory and experiment for rapid granular shear flow between identical boundaries. *Trans. ASME* E: *J. Appl. Mech.* (under review).

JENKINS, J. T. & RICHMAN, M. W. 1985 Kinetic theory for plane flows of a dense gas of identical, rough, inelastic circular disks. *Phys. Fluids* **28**, 3485–3495.

JENKINS, J. T. & RICHMAN, M. W. 1986 Boundary conditions for plane flows of smooth, nearly elastic, circular disks. *J. Fluid Mech.* **171**, 53–69.

JENKINS, J. T. & SAVAGE, S. B. 1983 A theory for the rapid flow of identical, smooth, nearly elastic particles. *J. Fluid Mech.* **130**, 187–202.

JOHNSON, P. C. & JACKSON, R. 1987 Frictional–collisional constitutive relations for granular materials with application to plane shearing. *J. Fluid Mech.* **176**, 67–93.

Louge, M. Y., Jenkins, J. T. & Hopkins, M. A. 1989 Computer simulation of rapid granular shear flows. *Phys. Fluids* A **2**, 1042–1044.

Richman, M. W. 1988 Boundary conditions based upon a modified Maxwellian distribution for flows of identical, smooth, nearly elastic spheres. *Acta Mech.* **75**, 227–240.

Richman, M. W. & Chou, C. S. 1988 Boundary effects on granular shear flows of smooth disks. *Z. Angew Math. Phys.* **39**, 885–901.

Walton, O. R. 1990 Computer simulation of rapidly flowing granular solids. In *Science on Form* (ed. S. Ishizaka). KTK Scientific Publishers.

Woodcock, L. V. 1981 Glass transition in the hard-sphere model and Kauzmann's paradox. *Ann. NY Acad. Sci.* **37**, 274–298.